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THE SEARCH FOR ONE AS A PRIME NUMBER: FROM ANCIENT GREECE TO MODERN TIMES

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Abstract. It has often been asked if one is a prime number, or if there was a time when most mathematicians thought one was prime. Whether or not the number one is prime is simply a matter of definition, but definitions are often decided by the use of mathematics. In this paper we will survey the history of the definition of prime as applied to the number one, from the ancient Greeks to the modern times. For the Greeks the numbers ($\alpha\rho\iota\theta\mu o\varsigma$) were multiples of the unit, and for this reason one did not fall into the category of primes (a subdivision of the numbers). This view held with few exceptions until Stevin (c. 1585) argued successfully that one was a number, at which point it finally made sense to ask if one is prime. This was followed by a period of confusion which *began* to dissipate with Gauss' *Disquisitiones Arithmeticae*. Our survey will show that for most of history, one was not considered a prime, and there was no point in time where a clear majority of mathematicians viewed one as prime.

1. Introduction

It has often been asked if the number one is prime, or if there was a time when most mathematicians thought one was prime. For example, the Online Encyclopedia of Integer Sequences has a sequence "Prime numbers at the beginning of the 20th century" which starts with one [74, Seq. A008578], and it is easy to find recent books written by non-mathematicians which list one as a prime (e.g., [3, p. 342], [16, p. 17] and [71, p. 76]).

Whether or not the number one is a prime is simply a matter of definition, but definitions are often decided by the use of mathematics, and we will see that as the uses of numbers changed so did views about the number one. In this paper we will survey the history of the primality of the number one, from the ancient Greeks to the modern times, along with some of the uses of primes which motivated the different definitions.

The Greeks were the first to study the prime numbers and put them to use in proving theorems. In the second section we will see that most of the ancient Greek scholars did not view one as a number (as an $\alpha\rho\iota\theta\mu o\varsigma$, or positive integer)—these numbers were *multiples* of the unit. For this reason, one did not fall into their category of primes (a subdivision of the numbers). As we discuss in the third

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section, this view held with few exceptions until Stevin (c. 1585) argued successfully that one was a number, and simultaneously laid the groundwork for the set of real numbers, combining Euclid's numbers ($\alpha\rho\iota\theta\mu o\varsigma$) with his magnitudes ($\mu\varepsilon\gamma\varepsilon\theta o\varsigma$).

In his excellent thesis "The Concept of One as a Number," C. Jones [44, p. 299] writes:

> In general, mathematics before Stevin is of one character and, after him, it is of another reflecting his contributions. In this regard, he is like Euclid: he stood at a watershed in the history of mathematics. And as with Euclid, he was so successful that, from our present day vantage point, it is hard to see the other side of that watershed. Over there, one was not a number; here and now, it is; even π is a number, and i, and aleph null.

In the fourth section we will discuss the period after one "becomes" a number (so it finally makes sense to ask if it is a prime) and before Gauss publishes his *Disquisitiones Arithmeticae*, which cemented unique factorization into primes as a key principle for the integers. In the fifth section, we will briefly mention how the discovery of number fields in general, and units in particular, sounded the death knell for one as a prime.

2. Greeks (600 BC–100 AD)

The Greeks were the first to formally study primes—so we begin with them. We will see that for most Greeks one was not a number, hence not a prime. In fact, for many the same holds for two.

According to Heath in his *History of Greek Mathematics,* "The first definition of number is attributed to Thales (c. 600 BC), who defined it as a collection of units" [38, p. 69]. Primes were defined as subsets of the numbers, so if one (the monad[1]) is excluded from the numbers, it cannot be prime. Aristotle (c. 384–322 BC) also "observes that the One is reasonably regarded as not being itself a number, because a measure is not the things measured,[2] but the measure of the One is the beginning of number" [38, p. 69].

The Pythagoreans (c. 500 BC) had a similar view of one, but endowed with deep mysticism:

> According to this principle the geometric figures and the numbers are divinely and inextricably linked. One, the monad, is not a number. It is the principle of 'sameness' of stability, right, equality, light. In eternal opposition is the dyad [two], the principle of 'otherness,' mutability, diversity, inequality, darkness. Itself not a number, it is the link between the monad and numbers. Three then is the first number in the Pythagorean sense, for as rennet curdles flowing milk by its peculiar creative and active faculty, so the unifying force of the monad advancing upon the dyad, source of every movement and breaking down, infixed a bound, and a form, that is number upon the triad; for this is the beginning of actual number. Three is the first number because number

[1] A monad, from the Greek $\mu o\nu\acute{\alpha}\varsigma$, is another word for a unit or unity.

[2] For the Greeks "a measures b" if $b = a + a + \ldots + a$ for some finite number of terms, e.g., $6 = 2{+}2{+}2$, so 2 measures 6; this is very similar to the modern notion of divides, but usually a number is not viewed as measuring itself.

> is limited by form and the triangle is the first plane figure. (Hopper [41, pp. 410-411])

Although Euclid (c. 300 BC) was not the first to define prime, for the most part his definitions stood for two thousand years. In the first two of these definitions from Euclid's *Elements* book VII [37, p. 277] Euclid defines the numbers ($\alpha\rho\iota\theta\mu o\varsigma$), and in the third he defines prime.

> Definition 1: A unit is that by virtue of which each of the things that exist is called one.
>
> Definition 2: A number is a multitude composed of units.
>
> Definition 11: A prime number is that which is measured by a unit alone.

As with most of the other ancient writers, Euclid did not need to say explicitly that one was not a prime because primes were a subcategory of the numbers, and one was not a number.

Neo-Pythagoreans Nicomachus (c. 100 AD) and Iamblichus (c. 250–325 AD) expanded on much of Euclid's work. Not only did they exclude two from the prime numbers, but they also defined composite numbers, numbers prime to one another, and numbers composite to one another as *excluding all even numbers*; they made all these categories subdivisions of odd [38, p. 73]. (That is, Nicomachus and Iamblichus said two is a number, just not a prime number.) Euclid, however, along with Aristotle (c. 384–322 BC) and Theon of Smyrna (c. 100 AD), included two among the prime numbers [38, p. 73]. All of these excluded one from the primes. A rare exception from this practice was Speusippus (c. 350 BC) who considered both one and two to be numbers and to be prime [77, p. 264, 276].

It can now be seen that while the Greeks were among the first to define what it means to be a number and a prime, these definitions excluded one (and sometimes two) from these categories. As one was not considered to be a number by most Greek mathematicians, there was not even a question about its primality: one was not a number, therefore it was not a prime.

3. Until Stevin argues one is a number (100–1585)

In the period between the Greeks and when one is finally widely accepted as a number, the integers began to slowly lose their mysticism. This period ends as Simon Stevin shows how to unite Euclid's numbers ($\alpha\rho\iota\theta\mu o\varsigma$) and his magnitudes ($\mu\varepsilon\gamma\varepsilon\theta o\varsigma$) into the modern reals, and as one begins to gain wide acceptance as a number.

We will use several extended quotes from Martianus Capella (c. 400 AD) [76, pp. 285-6] to illustrate this period:

> We have briefly discussed the number comprising the first series, the deities assigned to them, and the virtue of each number. I shall now briefly indicate the nature of number itself, what relations number bear to each other, and what forms they represent. A number is a collection of monads or a multiple preceding from a monad and returning to a monad.

This correspondence between deities and the numbers, which continued in various forms for some time, is an example of how this period's use of number differed from the modern. This difference makes many of their statements seemed obtuse, though they are still recognizable. For example, Capella defined prime as follows.

> Numbers are called prime which can be divided by no number; they are seen to be not 'divisible' by the monad but 'composed' of it: take, for example, the numbers five, seven, eleven, thirteen, seventeen, and others like them. No number can divide these numbers into integers. So they are called 'prime,' since they arise from no number and are not divisible into equal proportions. Arising in themselves, they beget other numbers from themselves, since even numbers are begotten from odd numbers, but an odd number cannot be begotten from even numbers. Therefore prime numbers must of necessity be regarded as beautiful.

Again, we see one is not considered prime as it is not a number.

> Let us consider all numbers of the first series according to the above classifications: the monad is not a number; the dyad is an even number; the triad is a prime number, both in order and in properties; the tetrad belongs in the even times even class; the pentad is prime; the hexad belongs to the odd times even or even times odd (hence it is called perfect); the heptad is prime; . . .

Capella's view (like Nicomachus' (c. 100) before him) that primes are a subset of the odds (so three was the first prime) was shared by many others: for example Boethius (c. 500) [58, pp. 89-95], Cassiodorus (c. 550) [35, p. 5], Isidore of Seville (c. 636) [35, pp. 4-5], and Hugh of St. Victor (c. 1120) [35, p. 56].

Most of the rest just omitted one from our modern list of primes, since primes were a category of number, and as J. Kobel writes in 1537 [59, p. 20]:

> Darauss verstehstu das 1. kein zal ist / sonder es ist ein gebererin / anfang / und fundament aller anderer zahlen.

That is,

> Wherefrom thou understandest that 1 is no number / but it is a generatrix / beginning / and foundation for all other numbers.

For example, al-Kindī (c. 850) not only argued at great length that one is not a number, but concluded further that it is neither even nor odd [46]. Since most mathematicians at this time still follow the Greeks, removing one as a number, why should writers such as Kobel, Prosdocimo (c. 1483) [75, pp. 13-4], G. Zarlino (c. 1561) [85, p. 22], even bother asking if one should be prime? They just started their lists of primes at two, as did P. A. Cataldi [17, p. 40] in 1603, C. Clavius [23, p. 307] in 1611, M. Mersenne [60, pp. 298-9] in 1625, A. Metius in 1640 [61, pp. 43-4], M. Bettini in 1642 [10, p. 36], etc.

This period ended as Simon Stevin (1548–1620) successfully argues one is a number, and while at it, that Euclid's distinction between numbers and magnitude should end—which essentially begins the development of the modern real numbers as Katz and Katz note [45]:

> Stevin created the basis for modern decimal notation in his 1585 work *De Thiende* ("the art of tenths"). He argued that quantities such as square roots, irrational numbers, surds, negative numbers, etc., should all be treated as numbers and not distinguished as being different in nature.

Stevin did this by providing notation for real numbers written in base 10 and providing algorithms for their manipulation. This change in notation is fundamental to changing how numbers were viewed:

> I believe there is no symbolism before about 1600, apart from numerals and a few things that are better described as abbreviations-may be connected with a new fluency in arithmetised thinking, which itself may owe a lot to the popularisation of decimal fractions at the end of the 15th century. Stevin, for example, was a thorough-going arithmetiser: he published, in 1585, the first popularisation of decimal fractions in the West (both in Dutch, *De Thiende*, and French, *La Disme*); in 1594, he described an algorithm for finding the decimal expansion of the root of any polynomial, the same algorithm we find later in Cauchy's proof of the intermediate value theorem to which I referred above; and he argued vigorously for an arithmetical understanding of the *Elements*, including its notorious Book X. (Fowler [30, p. 732])

For an excellent discussion of Stevin's view of one see Jones' fine thesis "The concept of <u>one</u> as a number," which discusses it at great length [44]. What is important here for the present paper is that for the first time, after two thousand years of one being omitted from the primes, by default one is now a number so it is now reasonable to ask: "is one prime?"

4. Until Gauss' 'uniqueness' (1585–1800)

During this period, the mathematicians before Gauss generally used prime numbers for factorization, but not usually for *unique* factorization. In their history of the Fundamental Theorem of Arithmetic, Ağargün and Özkan write that at this time "prime factorization was not looked upon as something of interest in its own right, but as a means of finding divisors" [2, p. 211].

Similarly, Ağargün and Fletcher write [1]:

> . . . it is significant that Propositions VII.31 and VII.30 of the Elements lead immediately to their proofs although Euclid forbears to take these steps. The first known proof of existence is due to al-Farisi (died c.1320), but he did not go on to prove uniqueness, mainly because his interest was in the divisors of a number rather than the factorisation itself . . . And if a mathematician's interest is in greatest common divisors, or perfect numbers, or amicable numbers then the divisors are the crucial objects, whereas the prime factorisation is just a means to an end.

We will illustrate this with a couple of common uses: finding divisors in order to find amicable and perfect numbers, and calculating logarithms of large integers.

Amicable numbers are defined as "a pair of numbers of which each of them is mutually equal to the sum of all the aliquot parts of the other" [43, p. 104]. In modern notation we let $\sigma(n)$ be the sum of the positive divisors of n and say that two integers m, n are an amicable pair if they satisfy the following:

$$\sigma(m) = m + n = \sigma(n).$$

For example 220 and 284 are an amicable pair because

$$\begin{aligned}\sigma(220) &= 1+2+4+5+10+20+11+22+44+55+110+220 = 220+284 = 504\\ &= 1+2+4+71+142+284 = 220+284 = \sigma(284).\end{aligned}$$

These special numbers had always have a special meaning of friendship—hence their name—due to their unique properties. Dickson noted that in Genesis 32:14 Jacob sent Esau 200 ewes and 20 rams, which amount to 220 goats, and that Rau

Nachshon (c. 850) comments "This number 220 (of goats) is a hidden secret ... tried by the ancients in securing the love of kings and dignitaries" [26, p. 39].

In addition, writers of this period were also very interested in perfect numbers[3], which are "that, all whole aliquot parts added together, make the same number with the number whereof they are such parts" [19]. In the modern definition, an integer n is considered a perfect number if

$$\sigma(n) = 2n.$$

Thus 6 is a perfect number because

$$\sigma(6) = 1 + 2 + 3 + 6 = 2 \cdot 6.$$

Much like amicable numbers, perfect numbers also held great significance. For instance, St. Augustine (c. 354) in his book *The City of God* [5, p. 337] wrote:

> These works [God's] are recorded to have been completed in six days (the same day being six times repeated), because six is a perfect number—not because God required a protracted time, as if He could not at once create all things, which then should mark the course of time by the movements proper to them, but because the perfection of the works was signified by the number six.

However, to find examples of perfect and amicable numbers, they needed to find factorizations, but it did not matter whether or not the number one was prime. One could be ignored in factorization.

Aside from finding amicable and perfect numbers, mathematicians used factorization in computing logarithms. John Napier (1550) developed the logarithms so that the calculation with large numbers would be easier. For example, the logarithms have the wonderful property that

$$\log_b(ac) = \log_b(a) + \log_b(c).$$

Therefore, multiplication becomes addition. This made prime factorizations useful to astronomers and scientists of this period in computing logarithms of large numbers. James Glaisher (1848) describes this process as follows in *Factor Table to the Fourth Million* [33, p. 46]

> Since the logarithm of a number is equal to the sum of the logarithms of its factors, it follows that if we have a table of logarithms of numbers up to a number n, we can obtain from it by simple addition the logarithm of any number whose greatest factor does not exceed n. A factor table therefore indicates all the numbers included in its range whose logarithms are obtainable by simple addition, and it also affords a simple method of calculating the logarithms of the others; for if the logarithm of a number x be required, and if it be prime, or its prime factors be not all less than n, we may seek in the neighborhood of x for a number $x \pm a$ whose prime factors are all less than n, and deduce the logarithm of x from that of $x \pm a$ by means of the formula
>
> $$\log x = \log(x \pm a) \mp \frac{a}{x} + \frac{1}{2}\frac{a^2}{x^2} \mp \frac{1}{3}\frac{a^3}{x^3} + \&c.$$

[3] Euclid showed in his *Elements* Book IX that if $2^n - 1$ is prime, then $2^{n-1}(2^n - 1)$ is perfect (Proposition 36). Later Euler proved in his *Tractatus de numerorum doctrina capita sedicim, quae supersunt* that any even perfect number has this form [28].

In this period astronomers and mathematicians would have tables of factorizations and logarithms on their shelves, but the log table might not reach as far as needed. So to compute the logarithm of a larger number, they would look for composites near this number, or near a small multiple of it. Glaisher gave an example [33, p. 46] using the prime 43867, which was beyond the limits of many log tables. Since

$$90 \cdot 43867 = 3,948,030 = 17^2 \cdot 19 \cdot 719 + 1,$$

it follows then that

$$43867 = \frac{1}{90}(17^2 \cdot 19 \cdot 719 + 1)$$

so (solving Glaisher's equation for $\log(x+a)$ instead of $\log(x)$)

$$\log 43867 = 2\log 17 + \log 19 + \log 719 - \log 90 + \\ + \frac{1}{3,948,029} - \frac{1}{2}\frac{1}{(3,948,029)^2} + \frac{1}{3}\frac{1}{(3,948,029)^3} - \&c.$$

The log used here is the natural log which at that time was called the hyperbolic log.

Again, though prime factorizations are used throughout this period, these uses rarely required unique factorization, so the question of whether the number one is a prime did not matter, and no consensus on the definition was established. For example, among those who wrote that one was prime were Brancker and Pell[4] (1668) [66, p. 114], F. Wallis (1685) [79, p. 292], G. S. Krüger (1746) [50, p. 839], M. L. Willich (1759) [84, p. 831], N. Caille and K. Scherffer (1762) [15, p. 13], J. H. Lambert (1770) [51, p. 73], A. Felkel (1776) [29], E. Waring (1782) [80, p. 379], A. G. Rosell (1785) [70, p. 39], A. Bürja (1786) [14, p. 45], etc. On the other hand, those who defined two as the first prime included F. Schooten (1657) [73, pp. 393-403], C. Chales (1690) [18, p. 169], J. Ozanam (1691) [65, p. 27], F. Brunot (1723) [12, p. 3], J. Cortes (1724) [24, p. 7], C. R. Reyneau (1739) [69, p. 248], S. Horsley (1772) [42, p. 332], etc.

In general, in this period the primality of one was not a matter of concern. Since the uniqueness of a factorization was not important to the main uses, the number one could be called prime or not. Usually most mathematicians ignored one in factorization even by those who labeled it prime. W. Milne illustrated this approach in his 1892 schoolbook [62] in which he wrote one as prime [pp. 91-2] but then omits it from the prime factorization of 1008 [p. 95].

5. Modern times (1800–present)

The modern use of primes began when Gauss proved that a number can be factored *uniquely* into primes (in articles 13-16 of his *Disquisitiones Arithmeticae*) [32, p. 6]:

> Theorem 16. A composite number can be resolved into prime factors in only one way.

This completed (what we now call) the Fundamental Theorem of Arithmetic [25, pp. 1-2]:

[4] M. Bullynck notes [13, p. 7] "The authorship of this book has been a matter of debate, but it is by now certain that Rahn was a student of Pell in Zürich and mainly used Pell's lectures to write his book. Already in 1668 the book has therefore been known as Pell's *Algebra*, and the *Table of Incomposits* has likewise been known as Pell's Table, though Keller and Brancker, independently, calculated it."

Theorem 1 (Fundamental Theorem of Arithmetic)**.** *For each natural number n there is a unique factorization*

$$n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k},$$

where exponents a_i are positive integers and $p_1 < p_2 < \cdots < p_k$ are primes.

The introduction of uniqueness changed the way that prime numbers were used, for now Gauss has codified Dirichlet's view that "the prime numbers are the material from which all other numbers may be built" [27, p. 9].

At this point, if one is a prime number, then we must state our theorems very carefully or unique factorization will be violated, so uniqueness was central to the demise of the view of one as a prime number among mathematicians. However, we believe that there is indeed something deeper and more fundamental than uniqueness involved here. At about the same time of Gauss' unique factorization, integers were generalized in several ways, each of which requires the notion of prime to be redefined. For example, when we add the negative integers to the positive, should we consider -3 to be a prime? What about the Gaussian "integers" $a + bi$ where a and b are integers? In this set, which, if any, of $3, -3, 3i, -3i$ should be a prime?

These types of examples prompted three changes: first, those numbers which divide one are separated into a special category called **the units** (e.g., in the integers $\{1, -1\}$, in the Gaussian integers $\{i, -i\}$); these units are excluded from the primes. Second, a prime element became one which satisfied the property described in Euclid's lemma (Book VII, Prop. 30, Euclid's *Elements* [37, p. 331]), which says that if a prime divides a product of two numbers, then it divides at least one of them. Therefore, in a modern text, a prime might be defined as follows:

> Definition. A non-unit element a of an integral domain is prime if whenever a divides the product bc, either a divides b or a divides c.

Similarly, an element q is an irreducible element if q cannot be written as a product of two non-units. Third, the notion of prime element became generalized to the prime ideal. All we need to say about this third case is units are excluded from generating prime ideals (because they generate the whole domain). Therefore, by all three, one is not a prime in the modern sense—a sense that takes root in Gauss' work.

For the reasons given above, at about this time, omitting one from the primes starts to become the standard among mathematicians and expositors: G. S. Klügel (1808) [47, p. 282], P. Barlow (1811) [7, p. 54], M. Ohm (1834) [64, p. 140], A. Reynaud (1835) [68, pp. 48-9], P. L. Chebyshev (1854) [20, p. 51], E. Meissel (1870) [33, p. 34], H. Scheffler (1880) [72, p. 79], G. Wertheim (1887) [83, p. 20], E. Lucas (1891) [55, pp. 350-1], P. Bachmann (1893) [6, p. 135], E. Landau (1909) [52, p. 3], H. Mangoldt (1912) [56, p. 176], E. Hecke (1923) [39, p. 5], B. L. van der Waerden (1949) [78, p. 59], etc.

So is that it? Has it been settled since Gauss that the number one is not prime? No, there are at least two reasons people may still wish to call one prime.

First, some might define one as prime due to tradition. For example, since the days of Pell's famous tables (1668) [57], many table makers had labeled one as prime. Table makers that continued this tradition are J. H. Lambert (1770) [51, p. 73], J. G. Garnier (1818) [31, p. 86], E. Hinkley (1853) [40, p. 7], T. Bertrand (1863) [9, p. 342], J. Glaisher (1876) [33, p. 232], etc. The last to do this (and one of the last to create tables of primes essentially by hand) was D. N. Lehmer in

1914. He explains the thinking of these table makers in the very first paragraph of the preface to his widely distributed table [54]:

> The number one is certainly not composite in the same sense as the number six, and if it is ruled out of the list of primes it is necessary to create a particular class for this number alone.

Now (and even in his day) there is a category for the number one: the units, but D. N. Lehmer clung to table makers' tradition.

Secondly, some label one as prime because they take Euclid's definition out of context. If we ignore that Euclid did not consider one a number, and essentially transliterate, we would get: a prime is a number only divisible by itself and one. This is often done by expositors such as M. Kraitchik (1942) [48, p. 78], A. Beiler (1964) [8, p. 223], etc. and writers who present mathematics tangentially like O. Gregory (1825) [36, pp. 44-5], C. Aschenborn (1867) [4, p. 86], E. Brooks (1873) [11, p. 58], W. Milne (1892) [62, pp. 91-2], etc. Some others that labeled one as prime wrote for students and the general public so they often lagged behind or simplified definition. This includes: A. M. Chmel (1807) [21, p. 65], A. M. Legendre (1830) [53, p. 14], F. Minsinger (1832) [63, pp. 36-7], J. Weigl (1848) [82, p. 28], J. Ray (1866) [67, p. 50], K. Weierstrass (1876) [81, p. 391], L. Kronecker (1901) [49, p. 303], G. Chrystal (1904) [22, p. 38], etc.

Despite the fact that some people might define one to be a prime number, essentially all modern mathematicians exclude one from the primes.

6. Conclusion

Again, the primality of one is only a matter of definition. If we restrict our view to the positive integers and define the number one to be prime, it would be possible to change the theorems involving the use of primes. For example, the Fundamental Theorem of Arithmetic (Theorem 1) could be adjusted by adding the two characters "$1 <$" as follows:

Theorem 2. *For each natural number n there is a unique factorization*

$$n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k},$$

where exponents a_i are positive integers and $1 < p_1 < p_2 < \cdots < p_k$ are primes.

However, as we have seen, these adjustments become very difficult when we generalize to other number systems such as number fields. Thus modern usage requires that one be omitted from the list of primes, not just for convenience (as implied by many such as Gowers [34]), but because 1, like -1 and $\pm i$, is a unit, and therefore, fundamentally different from the other numbers.

In our brief survey we have shown that the claim "one used to be a prime," is false, and surprisingly, for much of history one was not even a number. We have also shown there is no evidence that viewing one as a prime was ever a majority position at any time, and that Euclid, Cataldi, Mersenne, Euler, Gauss, etc. all omitted one from the primes. On the other hand, we have seen that there were (and are) traditionalists and others who have held the opposite view for much of the last few hundred years.

In a future paper we hope to explore the question of who was the last major mathematician to view one as a prime.

References

[1] A. G. Ağargün and C. R. Fletcher, *The fundamental theorem of arithmetic dissected*, The Mathematical Gazette **81** (1997), no. 490, 53–57.

[2] A. G. Ağargün and E. M. Özkan, *A historical survey of the fundamental theorem of arithmetic*, Historia Math. **28** (2001), no. 3, 207–214. MR **1849798 (2002e:**11003)

[3] J. B. Andreasen, L. Spalding, and E. Ortiz, *FTCE: Elementary Education K-6*, Cliffs Notes, John Wiley & Sons, Hoboken, NJ, 2011.

[4] C. H. M. Aschenborn, *Lehrbuch der Arithmetik mit Einschluss der Algebra und der niederen Analysis*, Verlag der Königlichen Geheimen Oberhofbuchdruckerei (R. V. Decker), Berlin, 1867, available from `http://books.google.com/books?id=3Rw7AQAAIAAJ`.

[5] Augustine, *The City of God*, Hendrickson Publishers, March 2009, Translator: M. Dods; available from `http://books.google.com/books?id=IwGePQYd4l0C&printsec`.

[6] P. Bachmann, *Die analytische Zahlentheorie*, Versuch einer Gesammtdarstellung dieser Wissenschaft in ihren Haupttheilen, Teubner, Leipzig, 1894, available from `http://books.google.com/books?id=AEsNAAAAYAAJ`.

[7] P. Barlow, *An Elementary Investigation of the Theory of Numbers, with its Application to the Indeterminate and Diophantine Analysis, the Analytical and Geometrical Division of the Circle, and Several Other Curious Algebraical and Arithmetical Problems*, J. Johnson and Co., London, 1811, available from `http://ebooks.library.cornell.edu/cgi/t/text/text-idx?c=math;idno=05210001`.

[8] A. Beiler, *Recreations in the Theory of Numbers*, Dover, New York, 1964, partially available from `http://books.google.com/books?id=fJTifbYNOzUC`.

[9] J. Bertrand, *Traité d'arithmétique*, 3rd ed., L. Hachette et Cie, 1863, available from `http://books.google.com/books?id=mf9BAAAAcAAJ`.

[10] M. Bettini, *Appiaria universae philosophicae mathematicae, in quibus paradoxa; accessit ad finem Euclides applicatus*, vol. I. Apiarium undecimum in quo arithmetica paradoxa et arcana, Typis Jo. Baptistae Ferronii, 1642, available from `http://books.google.com/books?id=iGBDAAAAcAAJ` (volume I follows volume II in this version).

[11] E. Brooks, *The New Normal Mental Arithmetic: A Thorough and Complete Course by Analysis and Induction*, Christopher Sower Company, Philadelphia, 1873, available from `http://books.google.com/books?id=VRGwHNTbcu8C`.

[12] F. Brunot, *Traité de la science des nombres: divisé en deux livres, où L'on donne des principes d'arithmetique et d'algebre*, Chez Quillau & Desaint, 1723, available from `http://books.google.co.uk/books?id=TK02AAAAMAAJ`.

[13] M. Bullynck, *Factor tables 1657–1817, with notes on the birth of number theory*, Rev. Histoire Math. **16** (2010), no. 2, 141–224 (2011). MR 2797739

[14] A. Bürja, *Der selbstlernende Algebrist*, F. T. Lagarde, 1786, available from `http://books.google.co.uk/books?id=8a02AAAAMAAJ`.

[15] N. L. L. Caille and K. Scherffer, *Lectiones elementares mathemaicae: seu, elementa algebrae, et geometriae*, J. T. Trattner, 1762, available from `http://books.google.co.uk/books?id=MrhJAAAAMAAJ`.

[16] Carnegie Library of Pittsburgh, *The Handy Science Answer Book*, The Handy Answer Book Series, Visible Ink Press, 2011.

[17] P. A. Cataldi, *Trattato de numeri perfetti*, Presso gli Heredi di Giovanni Rossi, Bologna, 1603, available from `http://mathematica.sns.it/media/volumi/69/`.

[18] C. F. M. Chales, *Cursus seu mundus mathematicus*, Anissonios, Posuel & Rigaud, Lugduni, 1690, available from `http://books.google.com/books?id=83pnrQrhB_IC`.

[19] E. Chambers, *Cyclopaedia, or an Universal Dictionary of Arts and Sciences*, Printed for W. Innys, etc. London, 1743, available from `http://books.google.com/books?id=ciiIaFU06vMC`.

[20] P. L. Chebyshev, A. A. Markov, and N. Sonin, *Œuvres de P. L. Tchebychef*, vol. 1, Commissionaires de l'Académie impériale des sciences, 1899, available from `http://books.google.com/books?id=TGBtAAAAMAAJ`.

[21] A. M. Chmel, *Institutiones mathematicae*, C. Haslinger, 1807, available from `http://books.google.co.uk/books?id=_tA2AAAAMAAJ`.

[22] G. Chrystal, *Textbook of Algebra*, Adam and Chas. Black, London, 1904, available fom `http://djm.cc/library/Algebra_Elementary_Text-Book_Part_I_Chrystal_edited.pdf`.

[23] C. Clavius, *Opera mathematica*, vol. I, sumptibus Antonij Hierat, excudebat Reinhardus Eltz, 1611, available from `http://books.google.com/books?id=lGBDAAAAcAAJ`.

[24] J. Cortès, *Arithmetica practica muy util y necessaria para todo genero de tratantes y mercaderes*, por los herederos de Diego de Larumbe, 1724, available from `http://books.google.co.uk/books?id=4Y_4r-X8zhMC`.

[25] R. Crandal and C. Pomerance, *Prime Numbers: A Computational Perspective*, 2nd ed., Springer, 2005.

[26] L. E. Dickson, *History of the Theory of Numbers*, vol. 1, AMS Chelsea Publishing, 1999.

[27] P. G. L. Dirichlet, *Lectures on Number Theory*, History of Mathematics, vol. 16, American Mathematical Society, Providence, RI, 1999, edited and supplemented by R. Dedekind, translated from the 1863 German original and with an introduction by John Stillwell, partially available from `http://books.google.com/books?id=8h8aWmnp1k8C&printsec=`. MR **1710911 (2000e:**01045)

[28] L. Euler, *Tractatus de numerorum doctrina capita sedicim, quae supersunt*, Comentationes arithmeticae 2 (1849), 503–575, available from `http://eulerarchive.maa.org/pages/E792.html`.

[29] A. Felkel, *Tafel aller einfachen Factoren der durch 2, 3, 5 nicht theilbaren Zahlen von 1 bis 10 000 000*, vol. I. Theil. Enthaltend die Factoren von 1 bis 144000, Wien: von Ehelenschen, 1776, available from `http://www.europeana.eu/portal/record/09428/E1B1958C0CE15249884EE1D17D5F510E996B9E33.html`.

[30] David Fowler, *Dedekind's theorem:* $\sqrt{2} \times \sqrt{3} = \sqrt{6}$, Amer. Math. Monthly **99** (1992), no. 8, 725–733. MR **1186456 (93h:**01022)

[31] J. G. Garnier, *Traité d'arithmétique*, J. N. Houdin, 1818, available from `http://books.google.co.uk/books?id=PAYOAAAAQAAJ`.

[32] C. F. Gauss, *Disquisitiones arithmeticae*, V. Fleischer, Leipzig, 1801, The English translation (by A. A. Clarke) was published in 1966 by Yale University Press and reprinted later by Springer-Verlag.

[33] J. Glaisher, *Factor Table for the Fourth Million: Containing the Least Factor of Every Number Not Divisible by 2, 3, or 5 between 3,000,000 and 4,000,000*, Taylor and Francis, London, 1879.

[34] T. Gowers, *Mathematics: A Very Short Introduction*, Oxford University Press, 2002.

[35] E. Grant, *A Source Book in Medieval Science*, Harvard University Press, 1974.

[36] O. Gregory, *Mathematics for Practical Men: Being a Common-Place Book of Principles, Theorems, Rules, and Tables...*, Baldwin, Cradock, & Joy, 1825, available from `http://books.google.co.uk/books?id=1o8-AAAAYAAJ`.

[37] T. L. Heath, *The Thirteen Books of the Elements*, vol. 2, Dover Publications, New York, 1956.

[38] ______, *A History of Greek Mathematics: From Thales to Euclid*, vol. 1, Dover, 1981.

[39] E. Hecke, *Vorlesungen über die Theorie der algebraischen Zahlen*, Chelsea scientific books, American Mathematical Society, 1970, First published 1923. Available from `http://books.google.com/books?id=MteTdxHj2ekC`.

[40] E. Hinkley, *Tables of the Prime Numbers, and Prime Factors of Composite Numbers, from 1 to 100,000: With the Methods of Their Construction, and Examples of Their Use*, Printed for the author, Baltimore, 1853, available from `http://openlibrary.org/books/OL24168026M`.

[41] G. M. Hopper, *The ungenerated seven as an index to Pythagorean number theory*, Am. Math. Monthly **43** (1936), no. 7, 409–413.

[42] S. Horsley, *Κόσκινον Ερατοσθένους, or, the sieve of Eratosthenes, being an account of his method of finding all the prime numbers*, Philosophical Transactions (1683-1775), Published by: The Royal Society **62** (1772), 327–347, available from `http://www.jstor.org/stable/106053`.

[43] C. Hutton, *A Mathematical and Philosophical Dictionary: Containing an Explanation of the Terms...*, J. Davis, 1795, available from `http://books.google.com/books?id=035RAAAAYAAJ`.

[44] C. J. Jones, *The concept of one as a number*, Ph.D. thesis, University of Toronto, 1978.

[45] K. U. Katz and M. G. Katz, *Stevin numbers and reality*, available from `http://arxiv.org/abs/1107.3688v2.`, 2011.

[46] Kindī and A. L. Ivry, *Al-Kindī's Metaphysics: A Translation of Ya'qub ibn Isaḥāq al-Kindī's Treatise "On First Philosophy" (fī al-Falsafah al-ūlā)*, Studies in Islamic philosophy and science, State University of New York Press, 1974.
[47] G. S. Klügel, *Mathematisches Wörterbuch*, Der Reine Mathematik, Leipzig, 1808, Available fron http://books.google.com/books?id=9jgDAAAAQAAJ.
[48] M. Kraitchik, *Mathematical Recreations*, Dover, New York, 1953.
[49] L. Kronecker, *Vorlesungen über Mathematik*, vol. 2, pt 1, B. G. Teubner, 1901, available from http://archive.org/details/vorlesungenberm00krongoog.
[50] J. G. Krüger, *Gedancken von der Algebra, nebst den Primzahlen von 1 vis 1000000*, Luederwalds Buchhandlung, 1746, available from http://books.google.com/books?id=2Lg2AAAAMAAJ.
[51] J. H. Lambert, *Zusätze zu den logarithmischen Tabellen*, Johann Heinrich, 1770, available from http://books.google.com/books?id=SjgHAAAAcAAJ.
[52] E. Landau, *Handbuch der Lehre von der Verteilung der Primzahlen*, vol. 1, B. G. Teubner, Leipiz and Berlin, 1909, available from http://books.google.com/books?id=81m4AAAAIAAJ.
[53] A. M. Legendre, *Théorie des nombres*, 3rd ed., vol. 1, Firmin Didot Frères, Paris, 1830, available from http://books.google.com/books?id=BTIVAAAAQAAJ.
[54] D. N. Lehmer, *List of Prime Numbers from 1 to 10,006,721*, Carnegie Institution of Washington, 1914.
[55] E. Lucas, *Théorie des nombres*, Gauthier-Villars, Paris, 1891, Reprinted 1991 Jacques Gabay, Sceaux, available from http://archive.org/details/thoriedesnombre00lucagoog.
[56] H. Mangoldt, *Einführung in die höhere Mathematik: bd. Differentialrechnung*, S. Hirzel, 1912, available from http://books.google.com/books?id=ybRXAAAAYAAJ.
[57] F. Maseres, J. Bernoulli, and J. Wallis, *The Doctrine of Permutations and Combinations, Being an Essential and Fundamental Part of the Doctrine of Chances*, Francis Maseres, Sold by B. and J. White, Fleet-Street., London, 1795, Reprints Bruckner's table of Incomposites. Available fron http://books.google.com/books?id=B05LAAAAMAAJ.
[58] M. Masi, *Boethian Number Theory—A Translation of the De Institutione Aritmetica*, Studies in Classical Antiquity, vol. 6, Rodopi, New York, 1983, partially available at http://books.google.com/books?id=StVUEsGK2RkC&printsec=.
[59] K. Menninger, *Number Words and Number Symbols: A Cultural History of Numbers*, Dover, 1992 (reprinted from 1969 translation), available from http://books.google.com/books?id=BFJHzSIj2u0C.
[60] M. Mersenne, *La Vérité des sciences: contre les sceptiques ou pyrrhoniens*, chez Tovssainct dv Bray, Paris, M. DC. XXV., 1625, available from http://books.google.com/books?id=FF44pHy9x_sC.
[61] A. Metius, *Arithmetica et geometria*, Appresso Simon Rocca, Venetia, 1640, available from http://books.google.com/books?id=ZdFGAAAAcAAJ.
[62] W. J. Milne, *Standard Arithmetic*, American Book Company, New York, 1892, available from http://books.google.com/books?id=ppnxMePtA2oC.
[63] F. Minsinger, *Lehrbuch der Arithmetik und Algebra*, Karl Köllman'schen, Ausburg, 1832, available from http://books.google.com/books?id=drEoAAAAcAAJ.
[64] M. Ohm, *Die reine Elementar-Mathematik: zum Gebrauche an höhern technischen Lehranstalten, besonders aber an Gymnasien und zum Selbst-Unterrichte*, vol. 1, Jonas Verlags Buchandlung, 1834, available from http://books.google.com/books?id=xPkUAQAAMAAJ.
[65] J. Ozanam, *Dictionaire mathematique, ou Idée generale des mathematiques*, aux dépens des Huguetan, 1691, available from http://books.google.com/books?id=SoU-AAAAcAAJ.
[66] J. H Rahn, *An Introduction to Algebra Translated out of the High-Dutch into English by Thomas Brancker. Much Altered and Augmented by D. P[ell] Also a Table of Odd Numbers less than One Hundred Thousand, shewing those that are Incomposit and Resolving the rest into their Factors or Coefficients, &c.*, W. G. for Moses Pitt at the White-Hart in Little Britain, 1668.
[67] J. Ray, *Primary Elements of Algebra for Common Schools and Academies*, Van Antwerp, Brag & Co., Cincinnati & New York, 1866, available from http://www.archive.org/details/primaryxxelement00rayjrich.
[68] A. A. L. Reynaud, *Traité d'arithmétique*, Bachelier, 1835, available from http://books.google.com/books?id=6LWE9SsKTYgC.

[69] C. R. Reyneau, *La Science du calcul des grandeurs en general: ou, Les Elemens des mathematiques*, Chez F. Pitteri, 1739, available from http://books.google.com/books?id=68BNAAAAMAAJ.

[70] A. G. Rosell, *Instituciones matemáticas*, vol. I, En la Imprenta Real, 1785, available from http://books.google.co.uk/books?id=_t70xeaCHK4C.

[71] C. Sagan, *Contact*, Pocket Books, 1997.

[72] H. Scheffler, *Die polydimensionalen Grössen und die vollkommenen Primzahlen*, Friedrich Vieweg and Sohn, 1880, available from http://books.google.com/books?id=9u0GAAAAYAAJ.

[73] F. Schooten, *Exercitationes mathematicae*, Lugd. Bat.: ex officina J. Elsevirii, 1657, available from http://books.google.com/books?id=-aBGAAAAcAAJ.

[74] N. J. A. Sloane, *The online encyclopedia of integer sequences*, Entry A008578 "Prime numbers at the beginning of the 20th century (today 1 is no longer regarded as a prime)," viewed 26 January 2012.

[75] D. E. Smith and G. A. Plimpton, *Rara arithmetica: A Catalogue of the Arithmetics Written Before the Year MDCI*, vol. 2, Ginn and Company, 1908, available from http://books.google.com/books?id=N44AAAAAMAAJ.

[76] W. H. Stahl and E. L. Burge, *Martianus Capella and the Seven Liberal Arts*, vol. II: The Marriage of Philology and Mercury, Columbia University Press, 1992, partially available from http://books.google.com/books?id=nZ-Z9eI6dXwC&printsec=.

[77] L. Tarán, *Speusippus of Athens: A Critical Study with a Collection of the Related Texts and Commentary*, Philosophia antiqua, vol. 39, E. J. Brill, Leiden, 1981, Partially online at http://books.google.com/books?id=cUPXqSb7V1wC.

[78] B. L. van der Waerden, *Modern Algebra*, vol. 1, Frederick Ungar Pub. Co., 1949, translated by Fred Blum.

[79] J. Wallis, *De algebra tractatus, historicus & practicus; cum variis appendicibus; operum mathematicorum volumen alterum*, E Theatro Sheldoniano, 1693, available from http://books.google.com/books?id=EuzpN1t5SOcC.

[80] E. Waring, *Meditationes algebraicæ*, J. Archdeacon, 1782, available from http://books.google.com/books?id=1MNbAAAAQAAJ.

[81] K. Weierstrass, *Vorlesungen uber die theorie der abelschen transcendenten*, Mathematische Werke von Karl Weierstrass (G. Hettner and J. Knoblauch, eds.), vol. 4, Mayer & Müller, Berlin, 1902, available from http://books.google.com/books?id=EFptAAAAMAAJ.

[82] J. B. Weigl, *Lehrbuch der Rechenkunst zum Gebrauche an lateinischen und Gewerbsschulen u. zum Selbstunterricht*, J. E. von Seidel, 1848, available from http://books.google.com/books?id=ZWIoAAAAcAAJ.

[83] G. Wertheim, *Elemente der Zahlentheorie*, Cornell University Library historical math. monographs, B. G. Teubner, 1887, available from http://books.google.com/books?id=k70KAAAAYAAJ.

[84] M. L. Willich and G. F. Rees, *Gründliche Vorstellung der Reesischen allgemeinen Regel nebst deren Anwendung auf die üblichsten Rechnungsarten*, Georg L. Förster, Bremen & Göttingen, 1759, available from http://www.archive.org/details/michellorenzwil00reesgoog.

[85] G. Zarlino, *Le istitutioni harmoniche*, Francesco Senese, Venetia, 1561, available from http://books.google.com/books?id=WsVCAAAAcAAJ.

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